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NAVIGATION EXPERIMENT UTILIZING RELAY II SATELLITE

Prepared under Contract No. NAS 5-9503 *by*
TRW SYSTEMS, INC.
Redondo Beach, Calif.
for Goddard Space Flight Center

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NAVIGATION EXPERIMENT UTILIZING

RELAY II SATELLITE

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TABLE OF CONTENTS

LIST OF FIGURES

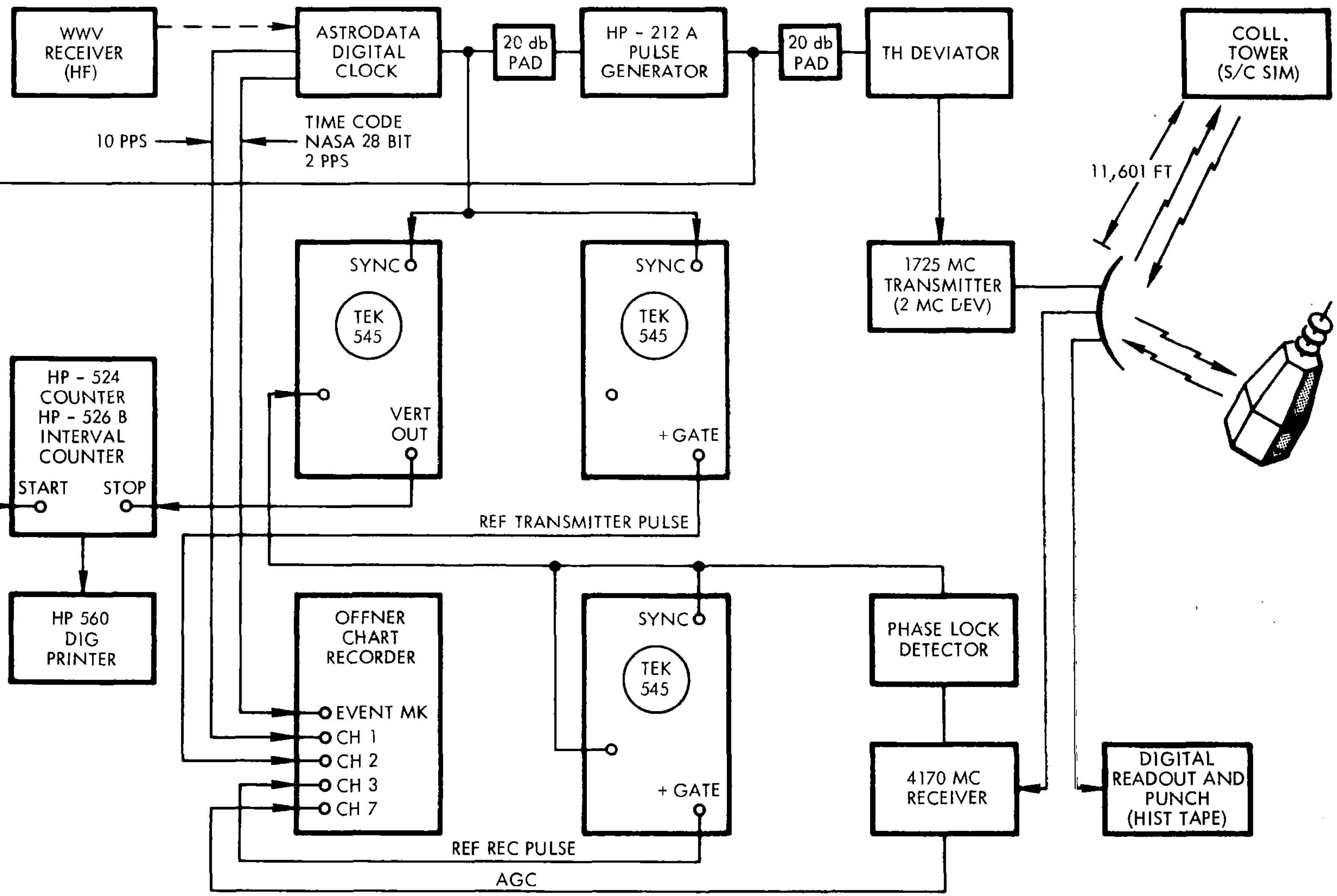

FIGURE 1. PATH DELAY TEST CONFIGURATION

3.0 REAL DATA ANALYSIS

The ranging measurements from the two runs (RELAY II Revs. 4250 and 4251) were processed by the TRW ESPOD orbit determination tracking program. This program performs a least square fit to the accurately known orbit and using a covariance matrix analysis to measure the errors as necessary, in the case of this experiment it was the station location. Refer to Appendix A for the detailed analysis. In actuality, an orbit was determined by using the range data and was then compared with the orbit supplied. The first run (Case 1) provided an overall error in station location of 2500 feet in East-West direction and 1300 feet in North-South. A measure of the actual total range uncertainty was also determined from the results. The mean value of range uncertainty (range measurement error) was found to be 1194 feet, however, the RMS value was 1761 feet.

The angle data taken during the two RELAY passes was not useful for reasons which are not known. The error analysis given in the following section shows that a considerable improvement in the station location measurement can be achieved if accurate azimuth and elevation angles (X and Y values in the case of Mojave) be measured along with the range data. The results of Case 1 were based on range measurements only and was workable because a very accurate estimate of the orbit was available. Meaningful results however could not be obtained from Case 2. A high sensitivity of the orbital parameters differential correction to the starting vector was apparent for the range only consideration for Case 2. This made it impossible to get an accurate solution.

4.0 ERROR ANALYSIS (See Appendix A)

The theoretical results were based on two cases that correspond to the actual data which was taken. The first case was 18.5 minutes of tracking with one range measurement every second. The second case was 12.0 minutes at the same data rate. In both cases range measurement uncertainties of 200, 500 and 1000 feet were assumed. It was also assumed that the uncertainty in the orbit parameters was much less than the station location uncertainties. The results are summarized in Figures 2, 3, 4 and 5. The uncertainties were expressed in units of distance in an up-meridian-parallel coordinate system.

For Case 1, the total vector uncertainty of station location which is the RMS of Figures 2, 3 and 4 for a range tracking sigma of 1000 feet is 1500 feet. For Case 2 however, it would require a range sigma of 150 feet for an uncertainty of 1500 feet. These results are summarized in Figure 5. The difference between Case 1 and 2 are seen to be an order of magnitude and this fact is borne out by considering Figure 6 where tracking uncertainty is plotted as a function of tracking time for Case 1 conditions and a range sigma of 500 feet. It is evident that station keeping ability is a very sensitive function of tracking time if a single pass is used.

This phenomenom is virtually independent of the number of points taken in the interval and it reflects the fact that an entire orbit must be reconstructed from a pass which covers only 10% of the central angle while tracking with only range data. In order to demonstrate this timing effect, a run was made with tracking from the station for using data from both passes. In other words, six minutes of total tracking means three minutes of the first pass (Case 1) and three minutes of the second (Case 2). The results have been plotted in Figure 7 and show a decrease in station uncertainty of almost 100:1 in some cases.

Finally, an analysis was made using azimuth and elevation data in addition to range. The results indicate a spectacular improvement in the possible accuracies. The run was made with range sigma equal to 500 feet and the angle noise equal to 18 seconds of arc. The results (in feet):

	σ_{NS}	σ_{EW}	σ_{ALT}	σ_{RAD}
Case 1	18.6	17.2	25.5	36.0
Case 2	15.7	14.3	21.9	30.5

In addition to the improvement in accuracy, a comparison of the 12 and 18 minute cases indicate that the sensitivity to tracking time has been significantly reduced. (For the range only case the ratio between Case 1 and Case 2 of σ_{RAD} is 6.6 while the angle data the ratio is .85). Case 2 is now slightly superior because the satellite spends proportionately more time higher in the sky.

5.0 CONCLUSIONS

The results of the actual data and error analysis lead to the conclusion that a station keeping system is quite feasible subject to the following recommendations and constraints:

a) Range, Azimuth and Elevation data be used.

b) The satellite orbit be known to great accuracy. Several unsuccessful runs have shown that the amount of data used in this study is not sufficient to generate orbital elements to any degree of accuracy. However, if several passes were available they could be used to update the orbital parameters before a station keeping run is made.

c) It is tacitly assumed that the system will not have any biases of the same order as the random noise components that were observed on the two measured passes. This is often not easy to achieve in field equipment and it must be remembered that the results will be degraded in proportion to the biases present.

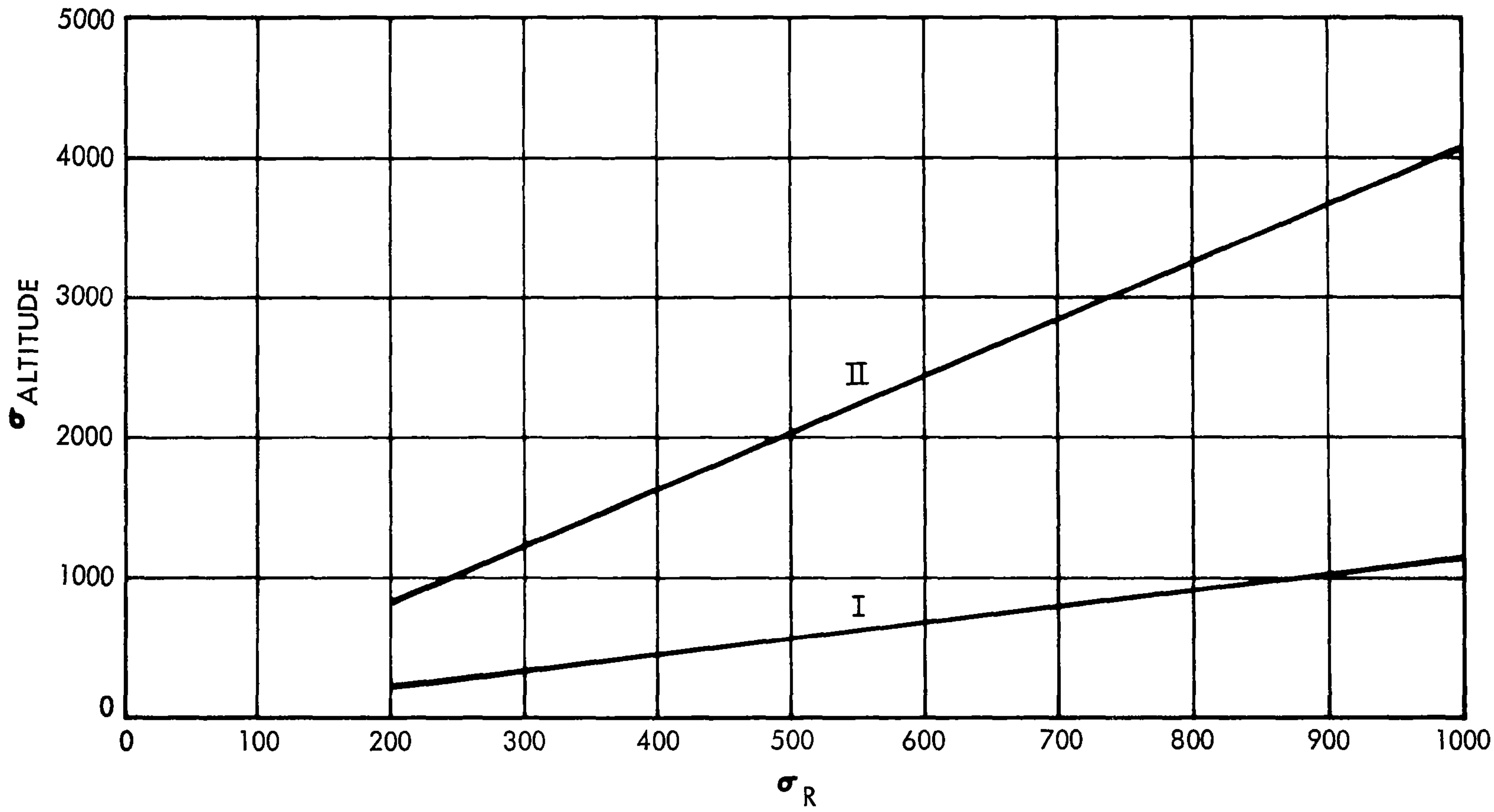

FIGURE 2. PREDICTED VALUE OF ALTITUDE UNCERTAINTY VS. RANGE UNCERTAINTY

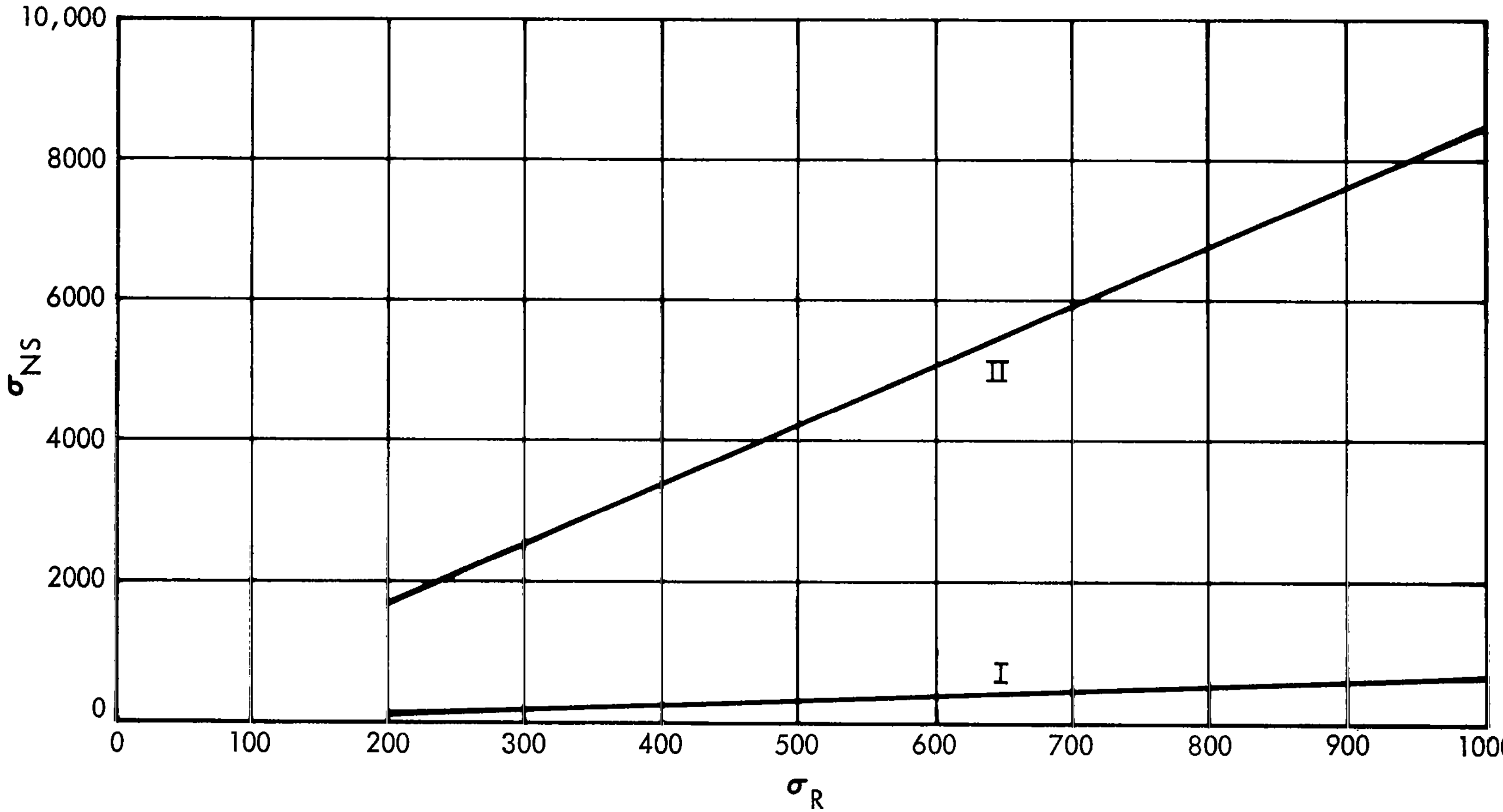

FIGURE 3. PREDICTED VALUE OF NORTH-SOUTH UNCERTAINTY VS. RANGE UNCERTAINTY

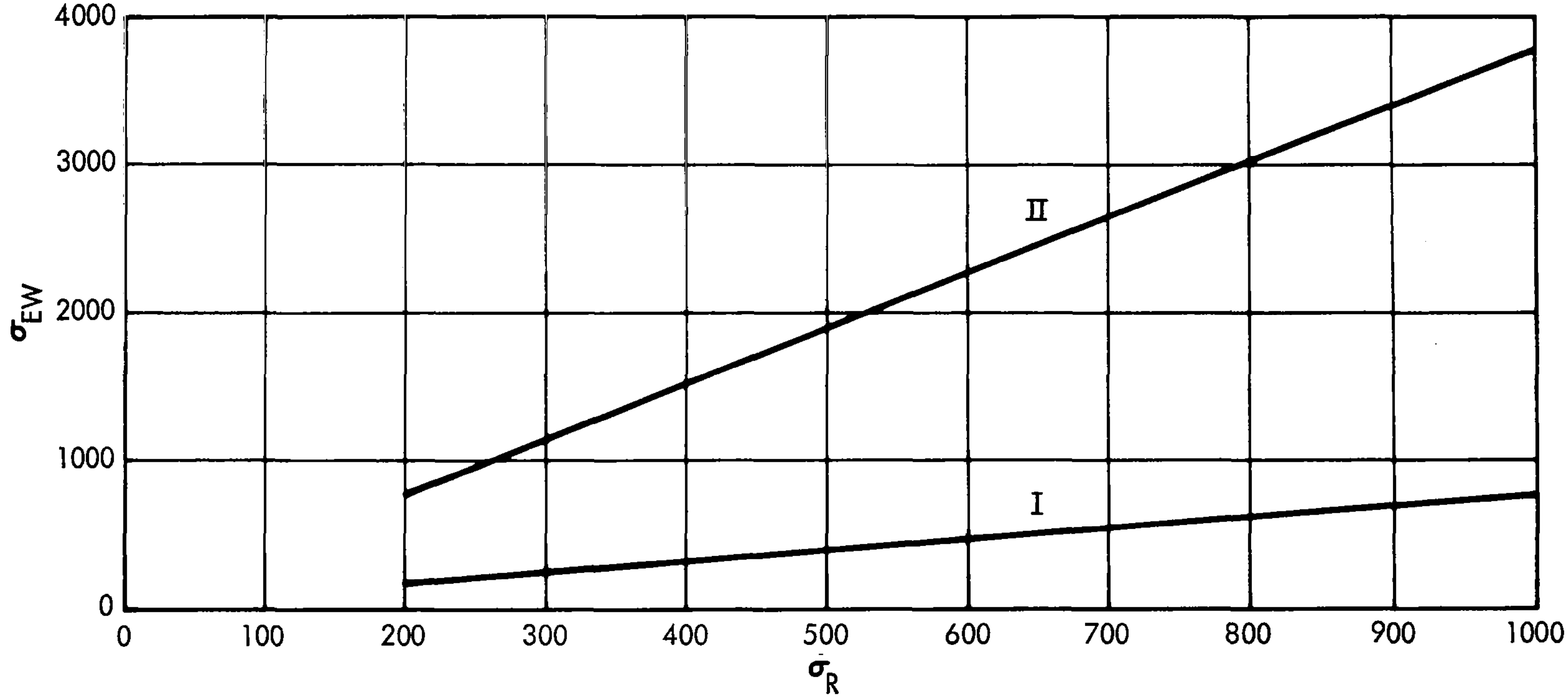

FIGURE 4. PREDICTED VALUE OF EAST-WEST UNCERTAINTY VS. RANGE UNCERTAINTY

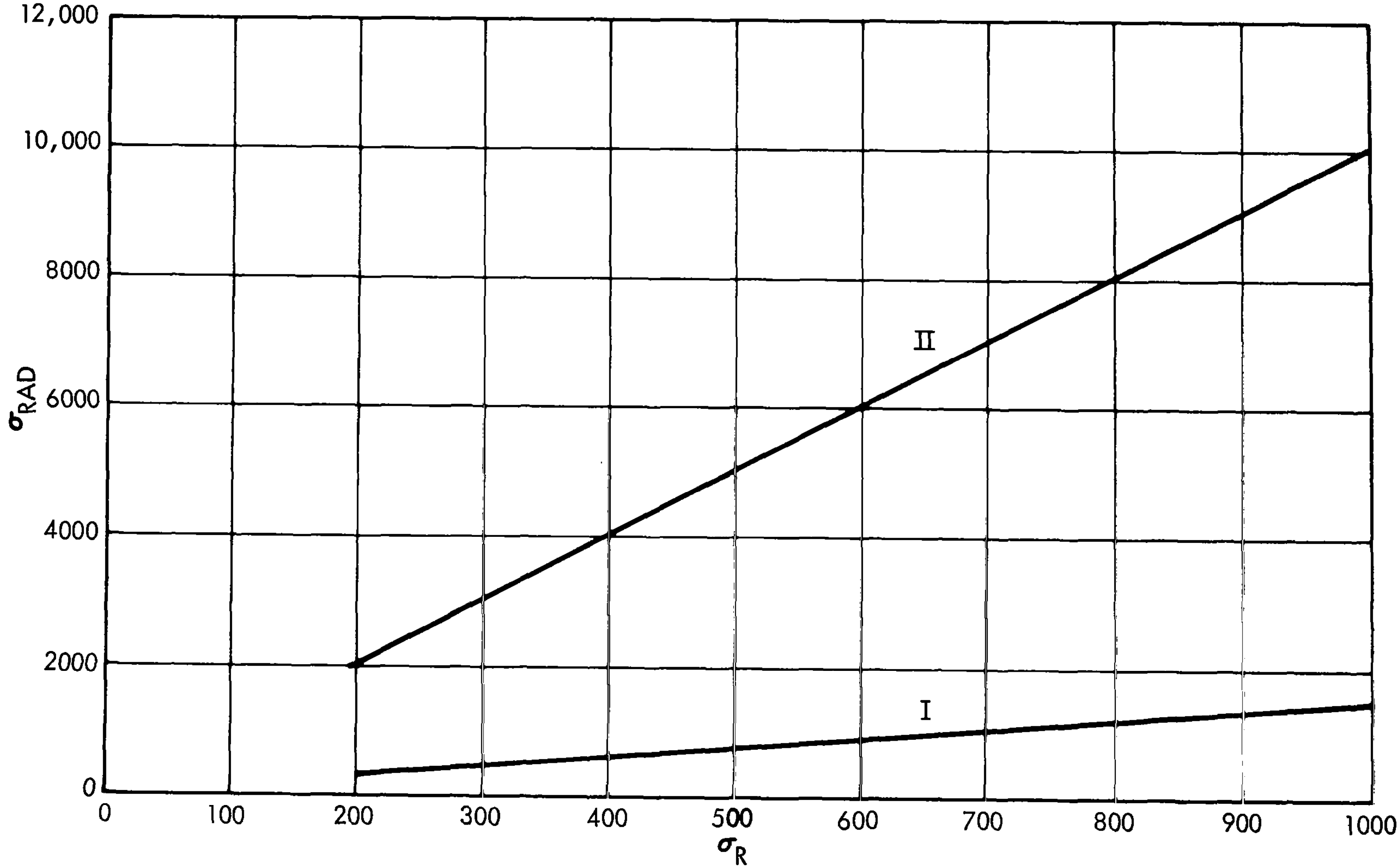

FIGURE 5. PREDICTED VALUE OF VECTOR MAGNITUDE UNCERTAINTY VS. RANGE UNCERTAINTY

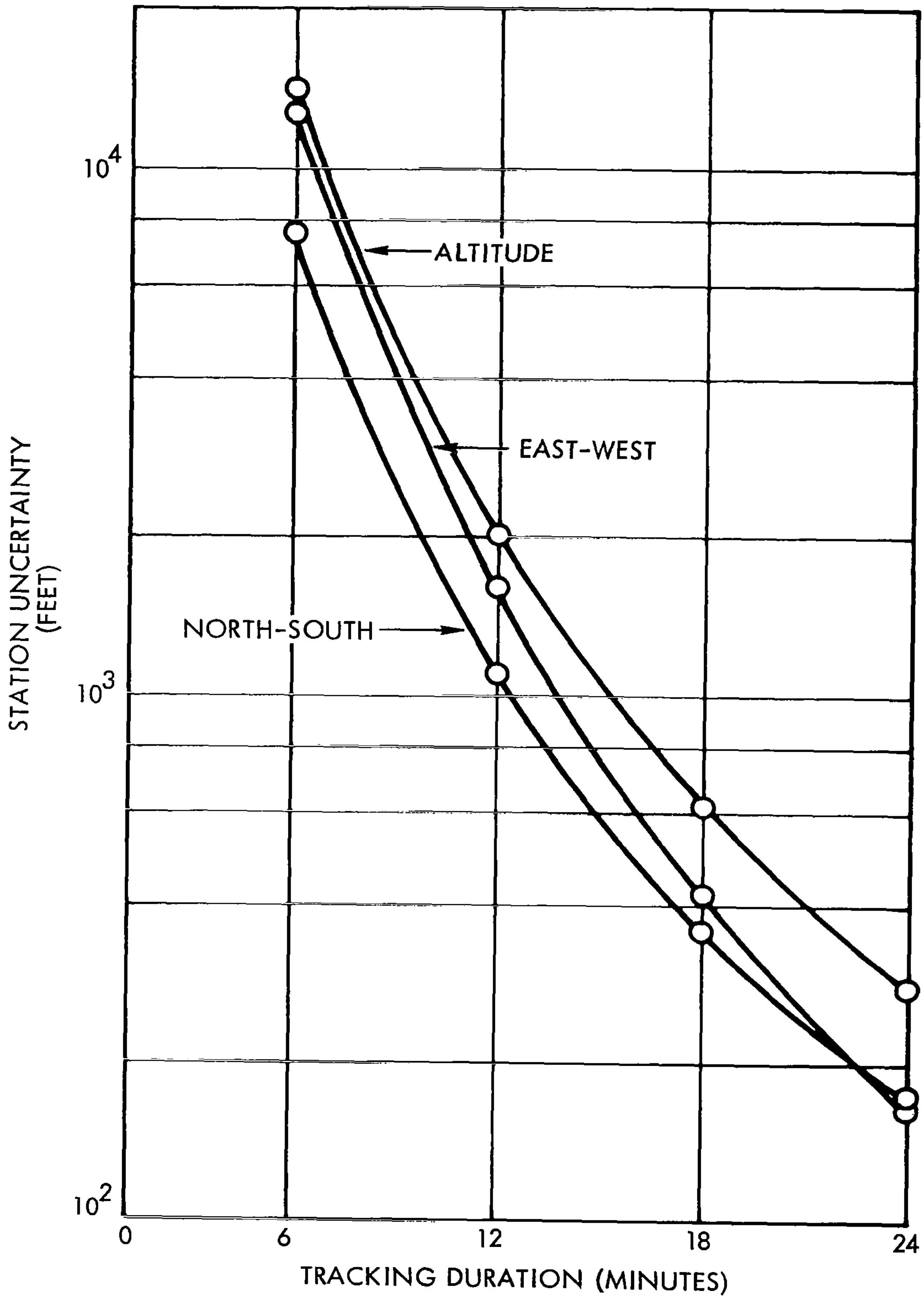

FIGURE 6. STATION UNCERTAINTY VS. DURATION OF TRACKING
Case I - Range σ = 500 ft.

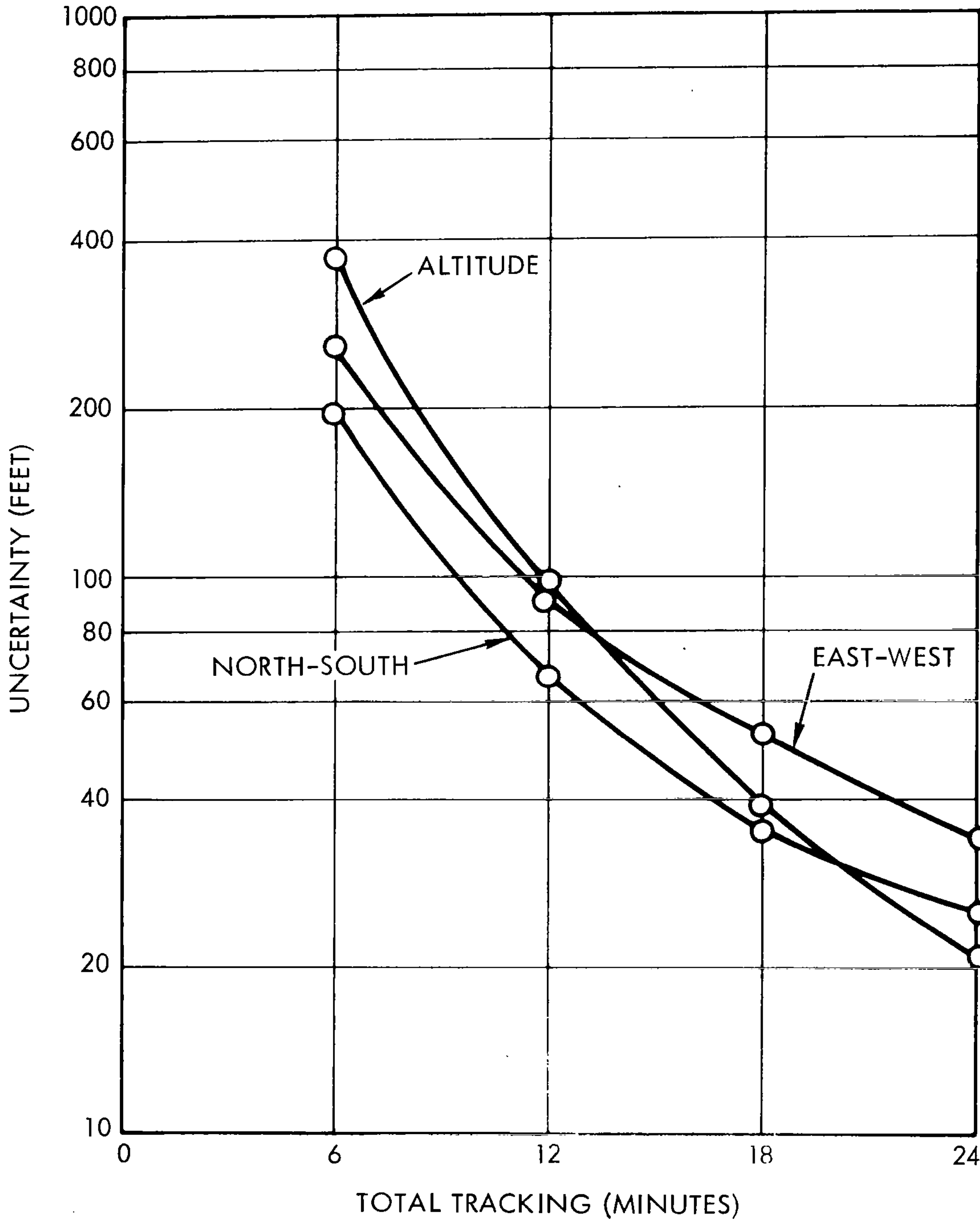

FIGURE 7. STATION UNCERTAINTY VS. TRACKING TIME USING TWO PASSES

APPENDIX A

ANALYSIS OF TRACKING ACCURACY

1.0 INTRODUCTION

The following introductory paragraphs are primarily a general review of the principles involved in the analysis. Symbol definitions are underlined.

Let η represent a random variable, which is defined as an ensemble of real-valued results of a repeatable experiment, or its sample value. In n dimensions η is an n-tuple $(\eta_1, \eta_2 \,.\,.\,.\,.\,.\,.\,.\, \eta_n)$ of real numbers η_i which, relative to a fixed basis, may also be expressed as the column vector:

$$\eta = \begin{bmatrix} \eta_1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_n \end{bmatrix}$$

Associated with η is a distribution (or cumulative distribution) function F(x) which is related to the probability of events defined in terms of the variable such that

$$\Pr(\eta_1 \leq x_1 \,.\,.\,.\,.\,.\,.\, \eta_n \leq x_n) = F(x_1 \,.\,.\,.\,.\, x_n)$$

For simplicity in notation, consider only the two dimensional case with continuous and independent random variables η_1 and η_2. Then there exists a <u>frequency function</u> $f(x_1, x_2)$ such that

$$f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2)$$

and

$$F(x_1, x_2) = \int_{-\infty}^{x_1} dt_1 \int_{-\infty}^{x_2} dt_2 \; f(t_1, t_2).$$

The <u>marginal frequency function</u> of η_1, which is defined as

$$Pr(\eta_1 \leq x_1) = Pr(\eta_1 \leq x_1, \; \eta_2 \leq \infty) = F(x_1, \infty).$$

is

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \, dx_2$$

The <u>conditional frequency function</u> which is defined as the probability of two events occurring simultaneously within specified intervals, of η_1 is

$$f(x_1 | x_2) = \frac{f(x_1, x_2)}{f_2(x_2)}$$

The marginal and conditional frequency functions for η_2 are similarly defined.

The mathematical expectation E of a function g of the variables is given by:

$$Eg(\eta_1, \eta_2) = \iint_{-\infty}^{\infty} g(x_1, x_2)\ f(x_1, x_2)\ dx_1\ dx_2$$

Let α represent the moment of the frequency function, which is defined as the expectation of powers of the random variables. That is,

$$\alpha^{pq} = E\eta_1^p\ \eta_2^q$$

From the above standard definitions and routine theory, it is readily verified that

$$\alpha^{10} = E\eta_1 = \mu_1$$

where μ_1 is the mean of the distribution relative to η_1. Similarly, μ_2 is the mean relative to η_2. Central moments are the expectations of powers of $(\eta - \mu)$. The second central moments $E(\eta-\mu)^2$ are the variances σ^2 (σ is the standard deviation).

In n dimension, the above definitions are obvious extensions of the two dimensional case. The second central moments become:

$$\sigma ij = E(\eta_i - \mu_i)(\eta_j - \mu_j)\ i, j = 1, 2, \ldots n.$$

and $E\eta_i = \mu_i$.

Note that $\sigma_{11} = \sigma_1^2$, $\sigma_{22} = \sigma_2^2$, etc. The symmetric matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$$

is the <u>covariance matrix</u>. A diagonal element is simply the variance of η_i. A non-diagonal element is the covariance of η_i and η_j, which are the cross products of deviations. Σ is non-negative definite and the rank of the matrix determines the character of the distribution.

In general, if A is a matrix with random elements, then $EA = (Ea_{ij})$. In vector/matrix notation,

$$\Sigma = E \begin{bmatrix} \eta_1 - \mu_1 \\ \vdots \\ \eta_n - \mu_n \end{bmatrix} \begin{bmatrix} (\eta_1-\mu_1) & \cdots & (\eta_n-\mu_n) \end{bmatrix}$$

$$= E\ (\eta-\mu)(\eta-\mu)'$$

Where the prime denotes the transpose matrix.

If Σ is multiplied from the left and from the right by the diagonal matrix of inverse standard deviations, the resultant symmetric matrix is the <u>correlation matrix</u> of $\eta_i \cdots \eta_n$:

$$\begin{bmatrix} 1 & \rho_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \rho_{1n} \\ \rho_{21} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \rho_{n1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

The elements of the correlation matrix are the correlation coefficients, which express the extent of the relation between the corresponding random variable η_i and η_j. If ρ_{ij} is zero, then η_i and η_j are independent. If $\rho_{ij} > 0$, the variables are positively related and if $\rho_{ij} < 0$, they are negatively related.

2.0 THE LEAST-SQUARES METHOD

2.1 Unweighted Least-Squares

The variable range R is a function of the random trajectory state variable X = (x, y, z). As a column vector,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Here, x, y, and z represent real values in the standard orthogonal coordinate system. (It should be noted that in the following general analysis X is not necessarily restricted to this designation nor to only three components. For example, $\dot{x}$, $\dot{y}$, and $\dot{z}$, could be included had range rate data been utilized).

Range is also a function f of X_o, an initial value of the state vector, and δX, a deviation from X_o:

$$R = f(X_o + \delta X).$$

Then, $$\delta R = \left.\frac{\partial f}{\partial X}\right|_{X = X_o} \delta X = A\delta X,$$

where A is the matrix $\left[\frac{x}{R}\ \frac{y}{R}\ \frac{z}{R}\right]$, since here $R = (x^2 + y^2 + z^2)^{1/2}$. (In vector notation $\delta\vec{R}_m = \vec{A}\delta X_A + \vec{n}$, where n is noise, which is considered later, the subscripts "m" denoting measured, and "A" actual).

In matrix notation, the sum S of the squares of the residuals is:

$$S = (\delta R - A\delta X)'(\delta R - A\delta X)$$

where the prime again denotes the matrix transpose.

The least-squares method requires that a value of δX, say $\delta\hat{X}$, be found such that $S(\delta\hat{X})$ is a minimum.

Since $(\delta R - A\delta X)' = \delta R' - (A\delta X)'$, then

$$dS = d\left[\delta R'\delta R - \delta R' A\delta X - (A\delta X)'\delta R + (A\delta X)'(A\delta X)\right]$$

By differentiation and equating dS to zero to solve for minimum S, it can be shown by several methods that the normal equation is

$$\delta\hat{X} = (A'A)^{-1} A'\delta R,$$

which is the minimum variance for an unweighted sum of squares.

2.2 Weighted Least-Squares

The only source of error is herein restricted to zero mean random noise. In this case,

$$\delta R = A\delta X + n$$

and $$C\delta R = CA\delta X + Cn$$

where C is a square matrix of order n X n. Let

$$C'C = W$$

where W is the noise weighting matrix. Note that W is symmetric and also non-negative definite.

Let
$$\delta R^* = C\delta R$$
$$A^* = CA$$
$$n^* = Cn$$

then $$\delta R^* = A^*\delta X + n^*.$$

Consider the least squares estimate of δX for the new regression equation:

$$\delta X = (A^{*\prime}A^*)^{-1}A^{*\prime}\delta R^*$$
$$= (A'C'CA)^{-1}A'C'C\delta R$$

Then $$\delta X = (A'WA)^{-1}A'W\delta R,$$

which is the <u>weighted least squares</u> estimate corresponding to the original regression equation. Ordinarily, the matrix C can be selected at the discretion of the analyst to take into account the various units used in the observations, or to reflect prior knowledge of the variance of the noise or even noise correlations.

2.3 Minimum Variance

The linear unbiased minimum variance estimate of the differential correction will be shown to be the weighted least squares estimate, where W is the inverse of the covariance matrix of the noise. By definition, an unbiased estimate is an estimate with a mean (or Expectation) equal to the parameter being estimated. Thus, it is required that

$$E\delta\hat{X} = \delta X$$

An estimate is called minimum variance if any other estimate has a larger covariance matrix (in the sense of positive definite).

Let a linear unbiased estimate of the deviation of the state vector be δX_E. Then the uncertainty in position is $\delta X_E - \delta X_A$, where δX_A represents the actual vector.

Let $\quad \delta X_E = B\delta R_A = B(A\delta X_A + n) = BA\delta X_A + Bn$

Then, $\quad E(\delta X_E) = \delta X_A = BA\delta X_A$

from the unbiased requirement and since $E(n) = 0$ from the zero mean noise assumption. So,

$$\delta X_E - \delta X_A = \delta X_E - BA\delta X_A = Bn$$

$$E\left[(\delta X_E - \delta X_A)(\delta X_E - \delta X_A)'\right] = BE(nn')\,B'$$

$$= BW^{-1}B'$$

where W^{-1} is the covariance matrix of the noise.

Suppose the first estimate $B_o = (A'WA)^{-1}A'W$.

Then, $\delta X_E = (A'WA)^{-1}A'W\delta R_A$

$= (A'WA)^{-1}A'W\ (A\delta X_A+n)$

$= \delta X_A + (A'WA)^{-1}A'Wn$

and, $E(\delta X_E) = \delta X_A$

Therefore, $B_o\delta R_A = \delta X_E$ is an unbiased estimate or the weighted least squares estimate is unbiased and has a covariance matrix:

$$\Sigma = B_o W^{-1} B_o'$$

$$= (A'WA)^{-1}A'WW^{-1}WA(A'WA)^{-1}$$

$$= (A'WA)^{-1}$$

consider any other unbiased estimate $\widetilde{\delta X}$.

$$\widetilde{\delta X} = B\delta R = (B_o + B_1)(A\delta X+n)$$

It can be shown that the covariance matrix of the new estimate is larger than Σ. Therefore, $\delta X_E = (A'WA)^{-1}A'W\delta R_A$ is a minimum variance unbiased linear estimate which is the weighted least squares estimate with W equal to the inverse of the noise covariance matrix.

$$A'WA = \frac{1}{\sigma^2}\begin{bmatrix} \frac{x^2}{R} & \frac{xy}{R} & \frac{xz}{R} \\ \frac{xy}{R} & \frac{y^2}{R} & \frac{yz}{R} \\ \frac{xz}{R} & \frac{yz}{R} & \frac{z^2}{R} \end{bmatrix}$$

3.0 COMBINATION OF TWO LEAST-SQUARES ESTIMATES

Consider two sets of data obtained from the same trajectory. For convenience, the first set can be considered as taken between times 1 and 2, and the second set between times 2 and 3. The residuals (before the fit) are given by

$$\delta R^{12} = A^{12}\,\delta X_A + n^{12}$$

and

$$\delta R^{23} = A^{23}\,\delta X_A + n^{23}$$

where δR^{ij} = the vector of residuals resulting from data taken from time i to time j $(= R_A{}^{ij} - R_R{}^{ij})$

$A^{ij} = \left[\frac{\partial R^{ij}}{\partial X}\right]$

δX_A = the vector deviation of the conditions at epoch of the actual trajectory from the conditions at epoch of the <u>reference</u> trajectory $(= X_A - X_R)$

n^{ij} = the vector of random noise on $R_A{}^{ij}$

The two least squares estimates are given by

$$\delta X_E{}^{12} = \left((A^{12})'\, W^{12} A^{12}\right)^{-1} (A^{12})'\; W^{12}\; \delta R^{12}$$

and

$$\delta X_E{}^{23} = \left((A^{23})'\, W^{23} A^{23}\right)^{-1} (A^{23})'\; W^{23}\; \delta R^{23}$$

If the residuals were combined into one vector, the least-squares estimate would be

$$\delta X_E^{13} = \left((A^{13})' W^{13} A^{13} \right)^{-1} (A^{13})' W^{13} \delta R^{13}$$

where $$\delta R^{13} = \begin{bmatrix} \delta R^{12} \\ \delta R^{23} \end{bmatrix}$$

$$A^{13} = \left[\frac{\partial R^{13}}{\partial X} \right] = \begin{bmatrix} \dfrac{\partial R^{12}}{\partial X} \\ \dfrac{\partial R^{23}}{\partial X} \end{bmatrix} = \begin{bmatrix} A^{12} \\ A^{23} \end{bmatrix}$$

If $$W^{13} = \begin{bmatrix} W^{12} & 0 \\ 0 & W^{23} \end{bmatrix}$$

Then δX_E^{13} can be reduced to:

$$\delta X_E^{13} = \left[(A^{12})' W^{12} A^{12} + (A^{23})' W^{23} A^{23} \right]^{-1}$$
$$\left[(A^{12})' W^{12} A^{12} \delta X_E^{12} + (A^{23})' W^{23} A^{23} \delta X_E^{23} \right]$$

This is the equation for combining two least-squares estimates under the above assumption about W^{13}.

4.0 UPDATING A LEAST-SQUARES ESTIMATE

The trajectory state vector X(t) can be thought of as a function of its value at any specified time and the time t. In particular

$$X(t) = f_1 \left[X(t_1),\ t \right] = f_2 \left[X(t_2),\ t \right]$$

The functions f_1 and f_2 describe the trajectory with epoch at t_1 and t_2, respectively.

Least squares estimates of the state vector at t_1 and at t_2 can be made from the same data. With epoch at t_1

$$\delta R = A_1 \, \delta X_{A1} + n$$

and $$\delta X_{E1} = (A_1' \, W A_1)^{-1} A_1' \, W \, \delta R$$

where $$A_i = \frac{\partial R}{\partial X(t_i)}$$

$$\delta X_{Ai} = \delta X_A(t_i)$$

$$\delta X_{Ei} = \text{estimate of } \delta X_A(t_i).$$

With epoch at t_2

$$\delta R = A_2 \, \delta X_{A2} + n$$

and $$\delta X_{E2} = (A_2' \, W A_2)^{-1} \, A_2' W \, \delta R$$

Now consider the relationship between A_1 and A_2.

$$A_2 = \frac{\partial R}{\partial X(t_2)} = \frac{\partial R}{\partial X(t_1)} \; \frac{\partial X(t_1)}{\partial X(t_2)} = A_1 \frac{\partial X_1}{\partial X_2}$$

Therefore

$$\delta X_{E2} = \left[\left(\frac{\partial X_1}{\partial X_2}\right)' A_1' \, W A_1 \frac{\partial X_1}{\partial X_2} \right]^{-1} \left(\frac{\partial X_1}{\partial X_2}\right)' \; A_1' W \, \delta R$$

or $$\delta X_{E2} = \frac{\partial X_2}{\partial X_1} \; (A'_1 \, W A_1)^{-1} \; A_1' W \, \delta R$$

since $$\left(\frac{\partial X_1}{\partial X_2}\right)^{-1} = \frac{\partial X_2}{\partial X_1}$$

Finally $$\delta X_{E2} = \frac{\partial X_2}{\partial X_1} \, \delta X_{E1}$$

which is the equation for updating a least-squares estimate from epoch 1 to epoch 2.

Next consider the error in the two estimates. Clearly

$$\delta X_{A2} = \frac{\partial X_2}{\partial X_1} \; \delta X_{A1}$$

Therefore

$$\delta X_{E2} - \delta X_{A2} = \frac{\partial X_2}{\partial X_1} \delta X_{E1} - \frac{\partial X_2}{\partial X_1} \delta X_{A1} = \frac{\partial X_2}{\partial X_1} (\delta X_{E1} - \delta X_{A1})$$

That is, the error is updated in the same manner as the estimate. Therefore,

$$\Sigma_2 = \frac{\partial X_2}{\partial X_1} \; \Sigma_1 \left(\frac{\partial X_2}{\partial X_1}\right)'$$

where $\Sigma_i = E\, (\delta X_{Ei} - \delta X_{Ai})(\delta X_{Ei} - \delta X_{Ai})'$